Test configurations and Okounkov bodies

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Abstract

We associate to a test configuration of an ample line bundle a filtration of the section ring of the line bundle. Using the recent work of Boucksom-Chen we get a concave function on the Okounkov body whose law with respect to Lebesgue measure determines the asymptotic distribution of the weights of the test configuration. We show that this is a generalization of a well-known result in toric geometry. As an application, we prove that the pushforward of the Lebesgue measure on the Okounkov body is equal to a Duistermaat-Heckman measure of a certain deformation of the manifold. Via the Duisteraat-Heckman formula, we get as a corollary that in the special case of an effective $\mathbb{C}^{\times}$-action on the manifold lifting to the line bundle, the pushforward of the Lebesgue measure on the Okounkov body is piecewise polynomial.

Contents

1 Introduction

1.1 Okounkov bodies

In [13] Okounkov introduced a way to associate a convex body in $\mathbb{R}^n$ to any ample divisor on a n-dimensional projective variety. This procedure was later shown to work in a more general setting by Lazarsfeld-Mustaţă in [11] and by Kaveh-Khovanskii in [8] and [9].

Let L be a big line bundle on a complex projective manifold X of dimension n. The Okounkov body of L, denoted by $\Delta(L)$, is a convex subset of $\mathbb{R}^n$, constructed in such a way so that the set-valued mapping

$$\Delta : L \longmapsto \Delta(L)$$

has some very nice properties (for the explicit construction see Section 2). It is homogeneous, i.e. for any $k \in \mathbb{N}$

$$\Delta(kL) = k\Delta(L).$$

Here kL denotes the the k:th tensor power of the line bundle L. Secondly, the mapping is convex, in the sense that for any big line bundles L and L', and any $k, m \in \mathbb{N}$, the following holds

$$\Delta(kL + mL') \supseteq k\Delta(L) + m\Delta(L'),$$

where the plus sign on the right hand side refers to Minkowski addition, i.e.

$$A + B := \{x + y : x \in A, y \in B\}.$$

Recall that the volume of a line bundle L, denoted by $\mathrm{vol}(L)$, is defined by

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{\dim H^0(kL)}{k^n/n!}.$$

By definition L is big if $\mathrm{vol}(L) > 0$. The third and crucial property, which makes Okounkov bodies useful as a tool in birational geometry, is that for any L

$$\mathrm{vol}(L) = n!\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)).$$

where the volume of the Okounkov body is measured with respect to the standard Lesbesgue measure on $\mathbb{R}^n$.

1.2 Test configurations

Given an ample line bundle L on X, a class of algebraic deformations of the pair (X, L), called test configurations, were introduced by Donaldson in [5], generalizing a previous notion of Tian [20] in the context of Fano manifolds. In short, a test configuration consists of:

(i) a scheme $\mathcal{X}$ with a $\mathbb{C}^\times$-action ρ,

(ii) an $\mathbb{C}^\times$-equivariant line bundle $\mathcal{L}$ over $\mathcal{X}$,

(iii) and a flat $\mathbb{C}^\times$-equivariant projection $\pi : \mathcal{X} \to \mathbb{C}$ such that $\mathcal{L}$ restricted to the fiber over 1 is isomorphic to L.

To a test configuration $\mathcal{T}$ there are associated discrete weight measures $\tilde{\mu}(\mathcal{T}, k)$ (see Section 4 for the definition). The asymptotics of the first moments of these measures are used to formulate stability conditions, such as K-stability, on the pair (X, L). These conditions are conjectured to be equivalent to the existence of a constant scalar curvature metric with Kähler form in $c_1(L)$, a conjecture which is sometimes called the Yau-Tian-Donaldson conjecture. This is one of the big open problems in Kähler geometry. By the works of e.g. Yau, Tian and Donaldson, a lot of progress has been made, in particular in the case of Kähler-Einstein metrics, i.e. when L is a multiple of the canonical bundle. For more on this, we refer the reader to the expository article [14] by Phong-Sturm.

When L is assumed to be a toric line bundle on a toric variety with associated polytope P, it was shown by Donaldson in [6] that a test configuration is equivalent to specifying a concave rationally piecewise affine function on the polytope P. This has made it possible to translate algebraic stability conditions on L into geometric conditions on P, which has proved very useful.

Heuristically, the relationship between a general line bundle L and its Okounkov body is supposed to mimic the relationship between a toric line bundle and its associated polytope. Therefore, one would hope that one could translate a general test configuration into some geometric data on the Okounkov body. The main goal of this article is to show that this in fact can be done, thus presenting a generalization of the well-known toric picture referred to above, and described in greater detail in Section 7.

1.3 The concave transform of a test configuration

By a filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$ we mean a vector space-valued map from $\mathbb{R} \times \mathbb{N}$,

$$\mathcal{F} : (t, k) \longmapsto \mathcal{F}_t H^0(kL),$$

such that for any k, $\mathcal{F}_t H^0(kL)$ is a family of subspaces of $H^0(kL)$ that is decreasing and left-continuous in t. $\mathcal{F}$ is said to be multiplicative if

$$(\mathcal{F}_t H^0(kL))(\mathcal{F}_s H^0(mL)) \subseteq \mathcal{F}_{t+s} H^0((k+m)L),$$

it is left-bounded if for all k

$$\mathcal{F}_{-t} H^0(kL) = H^0(kL) \qquad \text{for} \qquad t \gg 1,$$

and is said to linearly right-bounded if there exist a C such that

$$\mathcal{F}_t H^0(kL) = \{0\} \qquad \text{for} \qquad t \geq Ck.$$

The filtration $\mathcal{F}$ is called admissible if it has all the above properties.

Given a filtration $\mathcal{F}$, one may associate discrete measures $\nu(\mathcal{F}, k)$ on $\mathbb{R}$ in the following way

$$\nu(\mathcal{F}, k) := \frac{1}{k^n} \frac{d}{dt}(-\dim \mathcal{F}_{tk} H^0(kL)),$$

where the differentiation is done in the sense of distributions.

In a recent preprint [2] Boucksom-Chen show how any admissible filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$ of L gives rise to a concave function $G[\mathcal{F}]$ on the Okounkov body $\Delta(L)$ of L. $G[\mathcal{F}]$ is called the concave transform of $\mathcal{F}$. The main result of [2], Theorem A, states that the discrete measures $\nu(\mathcal{F}, k)$ converge weakly as k tends to infinity to $G[\mathcal{F}]_* d\lambda_{|\Delta(L)}$, the push-forward of the Lebesgue measure on $\Delta(L)$ with respect to the concave transform of $\mathcal{F}$.

Let $\mathcal{T}$ be a test configuration on (X, L). Given a section $s \in H^0(kL)$, there is a unique invariant meromorphic extension to configuration scheme $\mathcal{X}$. Using the vanishing order of this extension along the central fiber of $\mathcal{X}$ we define a filtration of the section ring $\oplus_k H^0(kL)$, which we show has the property that for any k

$$\tilde{\mu}(\mathcal{T}, k) = \nu(\mathcal{F}, k).$$

We will denote the associated concave transform by $G[\mathcal{T}]$. Combined with Theorem A of [2] we thus get our first main result.

Theorem 1.1. *Given a test configuration $\mathcal{T}$ of L there is a concave function $G[\mathcal{T}]$ on the Okounkov body $\Delta(L)$ such that the measures $\tilde{\mu}(\mathcal{T}, k)$ converge weakly as k tends to infinity to the measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$.*

We embed our test configuration into $\mathbb{C}$ times a projective space $\mathbb{P}^N$, so that the associated action comes from a $\mathbb{C}^\times$-action on $\mathbb{P}^N$. This we can always do (see e.g. [18]). The manifold X lies embedded in $\mathbb{P}^N$, and we thus via the action get a family X_τ of subamnifolds. As τ tends to 0, X_τ converges in the sense of currents to an algebraic cycle $|X_0|$ (see [6]). We let ω_{FS} denote the Fubini-Study on $\mathbb{P}^N$. Restricted to X_τ the (n,n)-form $\omega_{FS}^n/n!$ defines a positive measure, that as τ goes to zero converges to a positive measure $d\mu_{FS}$, the Fubini-Study volume form on $|X_0|$. There is also a Hamiltonian function h for the S^1-action. Using a result of Donaldson in [6] and Theorem 1.1 we can relate this picture with the concave transform by the following Corollary.

Corollary 1.2. *Assume that we have embedded the test configuration $\mathcal{T}$ in some $\mathbb{P}^N \times \mathbb{C}$, let h denote the corresponding Hamiltonian and $d\mu_{FS}$ the positive measure on $|X_0|$ defined above. Then we have that*

$$h_* d\mu_{FS} = G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

If $|X_0|$ is a smooth manifold, on which the S^1-action is effective, the measure $h_* d\mu_{FS}$ is the sort of measure studied by Duistermaat-Heckman in [7]. They prove that such a Duistermaat-Heckman measure is piecewise polynomial, i.e. the distribution function with respect to Lebesgue measure on $\mathbb{R}$ is piecewise polynomial. For a product test configuration, $|X_0| \cong X$, therefore we can apply the result of Duistermaat-Heckman to get the following.

Corollary 1.3. *Assume that there is a $\mathcal{C}^\times$-action on X which lifts to L, and that the corresponding S^1-action is effective. If we denote the associated product test configuration by $\mathcal{T}$, the concave transform $G[\mathcal{T}]$ is such that the pushforward measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ is piecewise polynomial.*

We also consider the case of a product test configuration, which means that there is an algebraic $\mathbb{C}^\times$-action ρ on the pair (X, L). We let φ be a positive S^1-invariant metric on L. Using the action ρ, we get a geodesic ray φ_t of positive metrics on L such that $\varphi_1 = \varphi$. Let us denote the t derivative at the point one by $\dot{\varphi}$. It is a real-valued function on X. There is also a natural volume element, given by $dV_\varphi := (dd^c\varphi)^n/n!$. By the function $\dot{\varphi}/2$ we can push forward the measure dV_φ to a measure on $\mathbb{R}$, which we denote by μ_φ. This measure does not depend on the particular choice of positive S^1-invariant metric φ. In fact, we have the following.

Theorem 1.4. *If we denote the product test configuration by $\mathcal{T}$, and the corresponding concave transform by $G[\mathcal{T}]$, then for any positive S^1-invariant metric φ it holds that*

$$\mu_\varphi = G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

The proof uses Theorem 1.1 combined with the approach of Berndtsson in [1], but is simpler in nature.

Phong-Sturm have in their articles [14] and [16] shown that the pair of a test configuration $\mathcal{T}$ and a positive metric φ on L canonically determines a $C^{1,1}$ geodesic ray of positive metrics on L emanating from φ. We conjecture that the analogue of Theorem 1.4 is true also in that more general case.

1.4 Organization of the paper

The definition of Okounkov bodies and some fundamental results concerning them is in Section 2, using [11] by Lazarsfeld-Mustaţă as our main reference.

Section 3 is devoted to describing the setup, definitions and main results of the article [2] by Boucksom-Chen on the concave transform of filtrations.

Section 4 contains a brief introduction to test configurations, following mainly Donaldson in [5] and [6].

We discuss embeddings of test configurations in Section 5, and link it to certain Duistermaat-Heckman measures.

In Section 6 we show how to construct the associated filtration to a test configuration, and prove Theorem 1.1, Corollary 1.2 and Corollary 1.3.

Section 7 concerns toric test configurations. We show that what we have done is a generalization of the toric picture, by proving that in the toric case, the concave transform is identical to the function on the polytope considered by Donaldson in [5].

Relying on the work of Ross-Thomas in [17] and [18], we obtain in Section 8 an explicit description of the concave transforms corresponding to a special class of test configurations, namely those arising from a deformation to the normal cone with respect to some subscheme.

In Section 9 we study the case of product test configurations, and relate it to geodesic rays of positive hermitian metrics. Hence we prove Theorem 1.4.

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2 The Okounkov body of a line bundle

Let Γ be a subset of $\mathbb{N}^{n+1}$, and suppose that it is a semigroup with respect to vector addition, i.e. if α and β lie in Γ, then the sum $\alpha+\beta$ should also lie in Γ. We denote by $\Sigma(\Gamma)$ the closed convex cone in $\mathbb{R}^{n+1}$ spanned by Γ.

Definition 2.1. *The Okounkov body $\Delta(\Gamma)$ of Γ is defined by*

$$\Delta(\Gamma) := \{\alpha : (\alpha, 1) \in \Sigma(\Gamma)\} \subseteq \mathbb{R}^n.$$

Since by definition $\Sigma(\Gamma)$ is convex, and any slice of a convex body is itself convex, it follows that the Okounkov body $\Delta(\Gamma)$ is convex.

By $\Delta_k(\Gamma)$ we will denote the set

$$\Delta_k(\Gamma) := \{\alpha : (k\alpha, k) \in \Gamma\} \subseteq \mathbb{R}^n.$$

It is clear that for all non-negative k,

$$\Delta_k(\Gamma) \subseteq \Delta(\Gamma) \cap ((1/k)\mathbb{Z})^n.$$

We will explain the procedure, which is due to Okounkov (see [13]), of associating a semigroup to a big line bundle.

Let X be a complex compact projective manifold of dimension n, and L a holomorphic line bundle, which we will assume to be big. Suppose we have chosen a point p in X, and local holomorphic coordinates $z_1, ..., z_n$ centered at p, and let $e_p \in H^0(U, L)$ be a local trivialization of L around p. If we divide a section $s \in H^0(X, kL)$ by e_p^k we get a local holomorphic function. It has an unique represention as a convergent power series in the variables z_i,

$$\frac{s}{e_p^k} = \sum a_\alpha z^\alpha,$$

which for convenience we will simply write as

$$s = \sum a_\alpha z^\alpha.$$

We consider the lexicographic order on the multiindices α, and let $v(s)$ denote the smallest index α such that $a_\alpha \neq 0$.

Definition 2.2. *Let $\Gamma(L)$ denote the set*

$$\{(v(s), k) : s \in H^0(kL), k \in \mathbb{N}\} \subseteq \mathbb{N}^{n+1}.$$

It is a semigroup, since for $s \in H^0(kL)$ and $t \in H^0(mL)$

$$v(st) = v(s) + v(t).$$

The Okounkov body of L, denoted by $\Delta(L)$, is defined as the Okounkov body of the associated semigroup $\Gamma(L)$.

We write $\Delta_k(\Gamma(L))$ simply as $\Delta_k(L)$.

Remark 2.3. *Note that the Okounkov body $\Delta(L)$ of a line bundle L in fact depends on the choice of point p in X and local coordinates z_i. We will however supress this in the notation, writing $\Delta(L)$ instead of the perhaps more proper but cumbersome $\Delta(L, p, (z_i))$.*

From the article [11] by Lazarsfeld-Mustaţă we recall some results on Okounkov bodies of line bundles.

Lemma 2.4. *The number of points in $\Delta_k(L)$ is equal to the dimension of the vector space $H^0(kL)$.*

Lemma 2.5. *We have that*

$$\Delta(L) = \overline{\cup_{k=1}^{\infty} \Delta_k(L)}.$$

Lemma 2.6. *The Okounkov body $\Delta(L)$ of a big line bundle is a bounded hence compact convex body.*

Definition 2.7. *The volume of a line bundle L, denoted by vol(L), is defined by*

$$\mathit{vol}(L) := \limsup_{k\to\infty} \frac{\dim H^0(kL)}{k^n/n!}.$$

The most important property of the Okounkov body is its relation to the volume of the line bundle, described in the following theorem.

Theorem 2.8. *For any big line bundle it holds that*

$$\mathit{vol}(L) = n!\mathit{vol}_{\mathbb{R}^n}(\Delta(L)),$$

where the volume of the Okounkov body is measured with respect to the standard Lesbesgue measure on $\mathbb{R}^n$.

For the proof see [11].

3 The concave transform of a filtered linear series

In this section, we will follow Boucksom-Chen in [2].

First we recall what is meant by a filtration of a graded algebra.

Definition 3.1. *By a filtration $\mathcal{F}$ of a graded algebra $\oplus_k V_k$ we mean a vector space-valued map from $\mathbb{R} \times \mathbb{N}$,*

$$\mathcal{F} : (t, k) \longmapsto \mathcal{F}_t V_k,$$

such that for any k, $\mathcal{F}_t V_k$ is a family of subspaces of V_k that is decreasing and left-continuous in t.

In [2] Boucksom-Chen consider certain filtrations which behaves well with respect to the multiplicative structure of the algebra.

They give the following definition.

Definition 3.2. *Let $\mathcal{F}$ be a filtration of a graded algebra $\oplus_k V_k$. We shall say that*

(i) $\mathcal{F}$ is multiplicative if

$$(\mathcal{F}_t V_k)(\mathcal{F}_s V_m) \subseteq \mathcal{F}_{t+s} V_{k+m}$$

for all $k, m \in \mathbb{N}$ and $s, t \in \mathbb{R}$.

(ii) $\mathcal{F}$ is pointwise left-bounded if for each k $\mathcal{F}_t V_k = V_k$ for some t.

(iii) $\mathcal{F}$ is linearly right-bounded if there exist a constant C such that for all k, $\mathcal{F}_{kC} V_k = \{0\}$.

A filtration $\mathcal{F}$ is said to be admissible if it is multiplicative, pointwise left-bounded and linearly right-bounded.

Given a line bundle L on X, its section ring $\oplus_k H^0(kL)$ is a graded algebra.

Boucksom-Chen in [2] show how an admissible filtration on the section ring $\oplus_k H^0(kL)$ of a big line bundle L gives rise to a concave function on the Okounkov body $\Delta(L)$. We will review how this is done.

First let us define the following set

$$\Delta_{k,t}(L, \mathcal{F}) := \{v(s)/k : s \in \mathcal{F}_t H^0(kL)\} \subseteq \mathbb{R}^n,$$

where as before $v(s) = \alpha$ if locally

$$s = Cz^\alpha + \text{higher order terms},$$

C being some nonzero constant. From the definition it is clear that

$$\Delta_{k,t}(L, \mathcal{F}) \subseteq \Delta_k(L),$$

since

$$\Delta_k(L) = \{v(s)/k : s \in H^0(kL)\}$$

and $\mathcal{F}_t H^0(kL) \subseteq H^0(kL)$. Similarly as in Lemma 2.4, from [11] we get that

$$|\Delta_{k,t}(L, \mathcal{F})| = \dim \mathcal{F}_t H^0(kL), \tag{1}$$

where $|.|$ denotes the cardinality of the set.

For each k we may define a function G_k on $\Delta_k(L)$ by letting

$$G_k(\alpha) := \sup\{t : \alpha \in \Delta_{k,t}(L, \mathcal{F})\}.$$

From the assumption that $\mathcal{F}$ is both left- and right-bounded it follows that G_k is well-defined and real-valued.

Lemma 3.3. *If we denote by $\nu_k(L)$ the sum of dirac measures at the points of $\Delta_k(L)$, i.e.*

$$\nu_k(L) := \sum_{\alpha \in \Delta_k(L)} \delta_\alpha,$$

then we have that

$$G_{k*}\nu_k(L) = \frac{d}{dt}(-\dim \mathcal{F}_t H^0(kL)).$$

Proof. By equation (1) and the definition of G_k we have that

$$\dim \mathcal{F}_t H^0(kL) = |\Delta_{k,t}(L,\mathcal{F})| = \int_{\{G_k \geq t\}} d\nu_k(L) = \int_t^{\infty} (G_k)_*(\nu_k(L)). \tag{2}$$

The lemma now follows by differentiating the equation (2). □

On the union $\cup_{k=1}^{\infty} \Delta_k(L)$ one may define the function

$$G[\mathcal{F}](\alpha) := \sup\{G_k(\alpha)/k : \alpha \in \Delta_k(L)\}.$$

By Boucksom-Chen in [2], or Witt Nyström in [21], one then gets that the function $G[\mathcal{F}]$ extends to a concave and therefore continuous function on the interior of $\Delta(L)$. In fact one gets that $G[\mathcal{F}]$ is not only the supremum but also the limit of G_k/k, i.e. for any $p \in \Delta(L)^\circ$

$$G[\mathcal{F}](p) = \lim_{k\to\infty} G_k(\alpha_k)/k,$$

for any sequence α_k converging to p.

Remark 3.4. *To show how this fits into the framework of [21], we note that if we let*

$$\tilde{G}(\alpha, k) := G_k(\alpha/k),$$

then $\tilde{G}$ is a function on $\Gamma(L)$. By the multiplicity of $\mathcal{F}$ it follows that $\tilde{G}$ is superadditive, and by the linear right-boundedness, $\tilde{G}$ is going to be linearly bounded from above. Thus one may apply the results of [21] to this function.

The main result of [2], Theorem A, is that we also have weak convergence of measures.

Theorem 3.5. *The measures*

$$\frac{1}{k^n}((G_k/k)_*\nu_k(L))$$

converge weakly to the measure

$$G[\mathcal{F}]_* d\lambda_{|\Delta(L)}$$

as k tends to infinity, where $d\lambda_{|\Delta(L)}$ denotes the Lebesgue measure on $\mathbb{R}^n$ restricted to $\Delta(L)$.

4 Test configurations

We will give a very brief introduction to the subject of test configurations. Our main references are the articles [5] and [6] by Donaldson.

First the definition of a test configuration, as introduced by Donaldson in [5].

Definition 4.1. *A test configuration $\mathcal{T}$ for an ample line bundle L over X consists of:*

(i) a scheme $\mathcal{X}$ with a $\mathbb{C}^\times$-action ρ,

(ii) an $\mathbb{C}^\times$-equivariant line bundle $\mathcal{L}$ over $\mathcal{X}$,

(iii) and a flat $\mathbb{C}^\times$-equivariant projection $\pi : \mathcal{X} \to \mathbb{C}$ where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication, such that $\mathcal{L}$ is relatively ample, and such that if we denote by $X_1 := \pi^{-1}(1)$, then $\mathcal{L}_{|X_1} \to X_1$ is isomorphic to $rL \to X$ for some $r > 0$.

By rescaling we can for our purposes without loss of generality assume that $r = 1$ in the definition.

A test configuration is called a product test configuration if there is a $\mathbb{C}^\times$-action ρ' on $L \to X$ such that $\mathcal{L} = L \times \mathbb{C}$ with ρ acting on L by ρ' and on $\mathbb{C}$ by multiplication. A test configuration is called trivial if it is a product test configuration with the action ρ' being the trivial $\mathbb{C}^\times$-action.

Since the zero-fiber $X_0 := \pi^{-1}(0)$ is invariant under the action ρ, we get an induced action on the space $H^0(kL_0)$, also denoted by ρ, where we have denoted the restriction of $\mathcal{L}$ to X_0 by L_0. Specifically, we let $\rho(\tau)$ act on a section $s \in H^0(kL_0)$ by

$$(\rho(\tau)(s))(x) := \rho(\tau)(s(\rho^{-1}(\tau)(x))). \tag{3}$$

Remark 4.2. *Some authors refer to the inverted variant*

$$(\rho(\tau)(s))(x) := \rho^{-1}(\tau)(s(\rho(\tau)(x)))$$

as the induced action. This is only a matter of convention, but one has to be aware that all the weights as defined below changes sign when changing from one convention to the other.

Any vector space V with a $\mathbb{C}^\times$-action can be split into weight spaces V_{η_i} on which $\rho(\tau)$ acts as multiplication by τ^{η_i}, (see e.g. [5]). The numbers η_i with non-trivial weight spaces are called the weights of the action. Thus we may write $H^0(kL_0)$ as

$$H^0(kL_0) = \oplus_\eta V_\eta$$

with respect to the induced action ρ.

In [14], Lemma 4, Phong-Sturm give the following linear bound on the absolute value of the weights.

Lemma 4.3. *Given a test configuration there is a constant C such that*

$$|\eta_i| < Ck$$

whenever $\dim V_{\eta_i} > 0$.

There is an associated weight measure on $\mathbb{R}$:

$$\mu(\mathcal{T}, k) := \sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_\eta,$$

and also the rescaled variant

$$\tilde{\mu}(\mathcal{T},k) := \frac{1}{k^n}\sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_{k^{-1}\eta}. \tag{4}$$

The first moment of the measure $\mu(\mathcal{T},k)$, which we will denote by w_k, thus equals the sum of the weights η_i with multiplicity $\dim V_{\eta_i}$. It can also be seen as the weight of the induced action on the top exterior power of $H^0(kL_0)$. The total mass of $\mu(\mathcal{T},k)$ is $\dim H^0(kL_0)$, which we will denote by d_k. By the flatness of π it follows that for k large it will be equal to $\dim H^0(kL)$ (see e.g. [17]). One is interested in the asymptotics of the weights, and from the equivariant Riemann-Roch theorem one gets that there is an asymptotic expansion in powers of k of the expression w_k/kd_k (see e.g. [5]),

$$\frac{w_k}{kd_k} = F_0 - k^{-1}F_1 + O(k^{-2}).$$

F_1 is called the Futaki invariant of $\mathcal{T}$, and will be denoted by $F(\mathcal{T})$.

Definition 4.4. *A line bundle L is called K-semistable if for all test configurations $\mathcal{T}$ of L over X, it holds that $F(\mathcal{T}) \geq 0$. L is called K-stable if it is K-semistable and furthermore $F(\mathcal{T}) = 0$ iff $\mathcal{T}$ is a product test configuration.*

Donaldson has conjectured that L being K-stable is equivalent to the existence of a positive constant scalar curvature hermitian metric with Kähler form in $c_1(L)$ (see [5], [6] and the expository article [15]).

5 Embeddings of test configurations

One way to construct a test configuration of a pair (X,L) is by using a Kodaira embedding of (X,L) into $(\mathbb{P}^N, \mathcal{O}(1))$ for some N. If ρ is a $\mathbb{C}^\times$-action on $\mathbb{P}^N$, this gives rise to a product test configuration of $(\mathbb{P}^N, \mathcal{O}(1))$. If we restrict to the image of ρ's action on (X,L), we end up with a test configuration of (X,L). A basic fact (see e.g. [18]) is that all test configurations arise this way, so that one may embed $\mathcal{X}$ into $\mathbb{P}^N \times \mathbb{C}$ for some N, the action ρ coming from a $\mathbb{C}^\times$-action on $\mathbb{P}^N$.

Let $\mathcal{T}$ be a test configuration, and assume that we have chosen an embeddding as above. Let z_i be homogeneous coordinates on $\mathbb{P}^N$, and let us define the following functions

$$h_{ij} := \frac{z_i \bar{z}_j}{||z||^2}.$$

We assume that we have chosen our coordinates so that the metric $||z||^2$ is invariant under the corresponding S^1-action on $\mathbb{C}^{N+1}$. Then the infinitesimal generator of the action ρ is given by a hermitian matrix A. We define a real-valued function h on $\mathbb{P}^N$ by

$$h := \sum A_{ij} h_{ij}.$$

It is a Hamiltonian for the S^1-action (see [6]). Let ω_{FS} denote the Fubini-Study form on $\mathbb{P}^N$. The zero-fiber X_0 of the test configuration can via the embedding be identified

with subsheme of $\mathbb{P}^N$, invariant under the action of ρ. By $|X_0|$ we will denote the corresponding algebraic cycle, and we let $[X_0]$ denote its integration current. The wedge product of $[X_0]$ with the positive (n,n)-form $\omega_{FS}^n/n!$ gives a positive measures, $d\mu_{FS}$, with $|X_0|$ as its support. We have the following proposition.

Proposition 5.1. *In the setting as above, the normalized weight measures $\tilde{\mu}(\mathcal{T},k)$ of the test configuration converges weakly as k tends to infinity to the pushforward of the measure $d\mu_{FS}$ with respect to the Hamiltonian h,*

$$\tilde{\mu}(\mathcal{T},k) \to h_* d\mu_{FS}.$$

Proof. This is essentially just a reformulation of a result by Donaldson in [6]. Using the weight measures $\tilde{\mu}(\mathcal{T},k)$, Equation (20) in the proof of Proposition 3 in [6] says that

$$\int_{\mathbb{R}} x^r d\tilde{\mu}(\mathcal{T},k) = \int_{|X_0|} h^r d\mu_{FS} + o(1).$$

for any positive integer r. In other words, for all such r, the r-moments of the measures $\tilde{\mu}(\mathcal{T},k)$ converge to the r-moment of the pushforward measure $h_* d\mu_{FS}$. But then it is classical that this implies weak convergence of measures. □

The measure $h_* d\mu_{FS}$ is the sort of measure studied by Duistermaat-Heckman in [7]. They consider a smooth symplectic manifold M with symplectic form σ, and an effective Hamiltonian torus action on M. This gives rise to a moment mapping J, which is a map from M to the dual of the Lie algebra of the torus, which we can naturally identify with $\mathbb{R}^k$, k being the dimension of the torus (we refer the reader to [7] for the definitions). There is a natural volume measure on M, given by $\sigma^n/n!$, called the Liouville measure. The pushforward of the Liouville measure with the moment map J, $J_*(\sigma^n/n!)$, is called a Duistermaat-Heckman measure. They prove that it is absolute continuous with respect to Lebesgue measure on $\mathbb{R}^k$, and provide an explicit formula, in the literature referred to as the Duistermaat-Heckman formula, for the density function f. As a corollary they get the following.

Theorem 5.2. *The density function f of the measure $J_*(\sigma^n/n!)$ is a polynomial of degree less that the dimension of M on each connected component of the set of regular values of the moment map J.*

In our setting the Liouville measure is given by $d\mu_{FS}$, and the moment map J is simply given by the Hamiltonian h. Thus when all components of the algebraic cycle $|X_0|$ are smooth manifolds, and the action is effective, we can apply Theorem 5.2 to our measure $h_* d\mu_{FS}$ and conclude that it is a piecewise polynomial measure on $\mathbb{R}$. In general of course some components of $|X_0|$ may have singularities. However, one case where we know that X_0 is a smooth manifolld is when we have a product test configuration, because then $X_0 = X$. Hence we get the following.

Proposition 5.3. *For a product test configuration, with a corresponding effective S^1-action, it holds that the law of the asymptotic distribution of its weights is piecewise polynomial.*

Proof. By Proposition 5.1 the law of the asymptotic distribution of weights is given by the measure $h_* d\mu_{FS}$ and by the remarks above we can use Theorem 5.2 to conclude that $h_* d\mu_{FS}$ is piecewise polynomial. $\square$

6 The concave transform of a test configuration

Given a test configuration $\mathcal{T}$ of L we will show how to get an associated filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$.

First note that the $\mathbb{C}^\times$-action ρ on $\mathcal{L}$ via the equation (3) gives rise to an induced action on $H^0(\mathcal{X}, k\mathcal{L})$ as well as $H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$, since $\mathcal{X} \setminus X_0$ is invariant.

Let $s \in H^0(kL)$ be a holomorphic section. Then using the $\mathbb{C}^\times$-action ρ we get a canonical extension $\bar{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$ which is invariant under the action ρ, simply by letting

$$\bar{s}(\rho(\tau)x) := \rho(\tau)s(x) \tag{5}$$

for any $\tau \in \mathbb{C}^\times$ and $x \in X$.

We identify the coordinate t with the projection function $\pi(x)$, and we also consider it as a section of the trivial bundle over $\mathcal{X}$. Exactly as for $H^0(\mathcal{X}, k\mathcal{L})$, ρ gives rise to an induced action on sections of the trivial bundle, using the same formula (3). We get that

$$(\rho(\tau)t)(x) = \rho(\tau)(t(\rho^{-1}(\tau)x) = \rho(\tau)(\tau^{-1}t(x)) = \tau^{-1}t(x), \tag{6}$$

where we used that ρ acts on the trivial bundle by multiplication on the t-coordinate. Thus

$$\rho(\tau)t = \tau^{-1}t,$$

which shows that the section t has weight -1.

By this it follows that for any section $s \in H^0(kL)$ and any integer η, we get a section $t^{-\eta}\bar{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$, which has weight η.

Lemma 6.1. *For any section $s \in H^0(kL)$ and any integer η the section $t^{-\eta}\bar{s}$ extends to a meromorphic section of $k\mathcal{L}$ over the whole of $\mathcal{X}$, which we also will denote by $t^{-\eta}\bar{s}$.*

Proof. It is equivalent to saying that for any section s there exists an integer η such that $t^\eta \bar{s}$ extends to a holomorphic section $S \in H^0(\mathcal{X}, k\mathcal{L})$. By flatness, which was assumed in the definition of a test configuration, the direct image bundle $\pi_*\mathcal{L}$ is in fact a vector bundle over $\mathbb{C}$. Thus it is trivial, since any vector bundle over $\mathbb{C}$ is trivial. Therefore there exists a global section $S' \in H^0(\mathcal{X}, k\mathcal{L})$ such that $s = S'_{|X}$. On the other hand, as for $H^0(kL_0)$, $H^0(\mathcal{X}, k\mathcal{L})$ may be decomposed as a direct sum of invariant subspaces $W_{\eta'}$ such that $\rho(\tau)$ restricted to $W_{\eta'}$ acts as multiplication by $\tau^{\eta'}$. Let us write

$$S' = \sum S'_{\eta'}, \tag{7}$$

where $S_{\eta'} \in W_{\eta'}$. Restricting the equation (7) to X gives a decomposition of s,

$$s = \sum s_{\eta'},$$

where $s_{\eta'} := S'_{\eta'|X}$. From (5) and the fact that $S'_{\eta'}$ lies in $W_{\eta'}$ we get that for $x \in X$ and $\tau \in \mathbb{C}^\times$ we have that

$$\bar{s}_{\eta'}(\rho(\tau)(x)) = \rho(\tau)(s_{\eta'}(x)) = \rho(\tau)(S'_{\eta'}(x)) = (\rho(\tau)S'_{\eta'})(\rho(\tau)(x))) = \\ = \tau^{\eta'}S'_{\eta'}(\rho(\tau)(x)),$$

and therefore $\bar{s}_{\eta'} = \tau^{\eta'}S'_{\eta'}$. Since trivially

$$\bar{s} = \sum \bar{s}_{\eta'}$$

it follows that $t^\eta \bar{s}$ extends holomorphically as long as $\eta \geq \max -\eta'$. □

Definition 6.2. *Given a test configuration $\mathcal{T}$ we define a vector space-valued map $\mathcal{F}$ from $\mathbb{Z} \times \mathbb{N}$ by letting*

$$(\eta, k) \longmapsto \{s \in H^0(kL) : t^{-\eta}\bar{s} \in H^0(\mathcal{X}, k\mathcal{L})\} =: \mathcal{F}_\eta H^0(kL).$$

It is immediate that $\mathcal{F}_\eta$ is decreasing since $H^0(\mathcal{X}, k\mathcal{L})$ is a $\mathbb{C}[t]$-module. We can extend $\mathcal{F}$ to a filtration by letting

$$\mathcal{F}_\eta H^0(kL) := \mathcal{F}_{\lceil \eta \rceil} H^0(kL)$$

for non-integers η, thus making $\mathcal{F}$ left-continuous. Since

$$t^{-(\eta+\eta')}\overline{ss'} = (t^{-\eta}\bar{s})(t^{-\eta'}\bar{s'}) \in H^0(\mathcal{X}, k\mathcal{L})H^0(\mathcal{X}, m\mathcal{L}) \subseteq H^0(\mathcal{X}, (k+m)\mathcal{L})$$

whenever $s \in \mathcal{F}_\eta H^0(kL)$ and $s' \in \mathcal{F}_{\eta'} H^0(kL)$, we see that

$$(\mathcal{F}_\eta H^0(kL))(\mathcal{F}_{\eta'} H^0(mL)) \subseteq \mathcal{F}_{\eta+\eta'} H^0((k+m)L),$$

i.e. $\mathcal{F}$ is multiplicative. Furthermore, by Lemma 6.1 it follows that $\mathcal{F}$ is left-bounded and right-bounded.

Proposition 6.3. *For $k \gg 0$*

$$\mu(\mathcal{T}, k) = \frac{d}{d\eta}(-\dim \mathcal{F}_\eta H^0(kL)).$$

Proof. Recall that we had the decomposition in weight spaces

$$H^0(kL_0) = \oplus_\eta V_\eta,$$

and that

$$\mu(\mathcal{T}, k) := \sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_\eta.$$

We have the following isomorphism:

$$(\pi_* k\mathcal{L})_{|\{0\}} \cong H^0(\mathcal{X}, k\mathcal{L})/tH^0(\mathcal{X}, k\mathcal{L}),$$

the right-to-left arrow being given by the restriction map, see e.g. [18]. Also, for $k \gg 0$, $(\pi_* k\mathcal{L})_{|\{0\}} = H^0(kL_0)$, therefore we get that for large k

$$H^0(kL_0) \cong H^0(\mathcal{X}, k\mathcal{L})/tH^0(\mathcal{X}, k\mathcal{L}), \tag{8}$$

We also had a decomposition of $H^0(\mathcal{X}, k\mathcal{L})$ into the sum of its invariant weight spaces W_η. By Lemma 6.1 it is clear that a section $S \in H^0(\mathcal{X}, k\mathcal{L})$ lies in W_η if and only if it can be written as $t^{-\eta}\bar{s}$ for some $s \in H^0(kL)$, in fact we have that $s = S_{|X}$. Thus we get that

$$W_\eta \cong \mathcal{F}_\eta H^0(kL),$$

and by the isomorphism (8) then

$$V_\eta \cong \mathcal{F}_\eta H^0(kL)/\mathcal{F}_{\eta+1} H^0(kL).$$

Thus we get

$$\dim \mathcal{F}_\eta H^0(kL) = \sum_{\eta' \geq \eta} \dim V_{\eta'}, \tag{9}$$

and the lemma follows by differentiating with respect to η on both sides of the equation (9). □

Proposition 6.4. *The filtration associated to a test configuration $\mathcal{T}$ is always admissible. If we let $G_k[\mathcal{T}]$ denote the functions on $\Delta_k(L)$ associated to the filtration $\mathcal{F}(\mathcal{T})$ as previously definied, then we have that*

$$\mu(\mathcal{T}, k) = G_k[\mathcal{T}]_* \nu_k(L) \tag{10}$$

and

$$\tilde{\mu}(\mathcal{T}, k) = \frac{1}{k^n}((G_k[\mathcal{T}]/k)_*(\nu_k(L))). \tag{11}$$

Proof. The equality of measures (10) follows immediately from combining Lemma 3.3 and Proposition 6.3, and (11) is just a rescaling of (10). Since by Lemma 4.3 the weights of a test configuration is linearly bounded, by (10) we get that the same holds for the functions $G_k[\mathcal{T}]$, i.e. the filtration $\mathcal{F}$ is linearly left- and right-bounded. It is hence admissible, since the other defining properties had already been checked. □

Theorem 6.5. *With the setting as in the proposition above, we have the following weak convergence of measures as k tends to infinity*

$$\tilde{\mu}(\mathcal{T}, k) \to G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

Proof. Follows from Theorem 3.5 together with Proposition 6.4. □

Corollary 6.6. *In the asymtotic expansion*

$$\frac{w_k}{kd_k} = F_0 - k^{-1}F_1 + O(k^{-2})$$

we have that

$$F_0 = \frac{n!}{vol(L)} \int_{\Delta(L)} G(\mathcal{T}) d\lambda.$$

Proof. Recall that in Section 4 we defined w_k by

$$w_k := \int_{\mathbb{R}} x d\mu(\mathcal{T}, k),$$

i.e. in other words

$$w_k = \sum \eta \dim V_\eta,$$

$\oplus_\eta V_\eta$ being the weight space decomposition of $H^0(kL_0)$.Thus Theorem 6.5 implies that

$$\lim_{k\to\infty} \frac{w_k}{k^{n+1}} = \lim_{k\to\infty} \int_{\mathbb{R}} x\tilde{\mu}(\mathcal{T}, k) = \int_{\mathbb{R}} x(G[\mathcal{T}])_*(d\lambda_{|\Delta(L)}) = \int_{\Delta(L)} G(\mathcal{T})d\lambda, \quad (12)$$

using the weak convergence and the definition of the push forward of a measure. (12) together with the standard expansion

$$d_k := \dim H^0(kL) = k^n vol(L)/n! + o(k^n)$$

yields the corollary. □

Another consequence of Theorem 6.5 is that it relates the Okounkov body $\Delta(L)$ with the central fibre X_0, and therefore X, in the sense of the following corollary.

Corollary 6.7. *Assume that we have embedded the test configuration $\mathcal{T}$ in some $\mathbb{P}^N \times \mathbb{C}$, let h denote the corresponding Hamiltonian and $d\mu_{FS}$ the Fubini-Study volume measure on $|X_0|$ as in Section 4. Then we have that*

$$G[\mathcal{T}]_* d\lambda_{|\Delta(L)} = h_* d\mu_{FS}.$$

Proof. Follows immediately from combining Proposition 5.1 and Theorem 6.5. □

As in Section 5, if restrict to the case of product test configurations where the S^1-action is effective, we can apply the Duistermaat-Heckman theorem to these measures, and get the following.

Corollary 6.8. *Assume that there is a $\mathcal{C}^\times$-action on X which lifts to L, and that the corresponding S^1-action is effective. If we denote the associated product test configuration by $\mathcal{T}$, the concave transform $G[\mathcal{T}]$ is such that the pushforward measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ is piecewise polynomial.*

Proof. Follows from combining Proposition 5.3 and Corollary 6.7. □

7 Toric test configurations

We will cite some basic facts of toric geometry, all of which can be found in the article [5] by Donaldson. Let $L_P \to X_P$ be a toric line bundle with corresponding polytope $P \subseteq \mathbb{R}^n$. Thus for every k there is a basis for $H^0(kL_P)$ such that there is a one-one

correspondence between the basis elements and the integer lattice points of kP. We write this as

$$\alpha \in kP \cap \mathbb{Z}^n \leftrightarrow z^\alpha \in H^0(kL_P).$$

In [5] Donaldson describes the relationship between toric test configurations and the geometry of polytopes. Let g be a positive concave rational piecewise affine function defined on P. One may define a polytope Q in $\mathbb{R}^{n+1}$ with P as its base and the graph of g as its roof, i.e.

$$Q := \{(x, y) : x \in P, y \in [0, g(x)]\}.$$

That g is rational means precisely that the polytope Q is rational, i.e. it is the convex hull of a finite set of rational points in $\mathbb{R}^n$. In fact, by scaling we can without loss of generality assume that Q is integral, i.e. the convex hull of a finite set of integer points. Then by standard toric geometry this polytope Q corresponds to a toric line bundle L_Q over a toric variety X_Q of dimension $n+1$. We may write the correspondence between integer lattice points of kQ and basis elements for $H^0(kL_Q)$ as

$$(\alpha, \eta) \in kQ \cap \mathbb{Z}^{n+1} \leftrightarrow t^{-\eta} z^\alpha \in H^0(kL_Q). \tag{13}$$

There is a natural $\mathbb{C}^\times$-action ρ given by multiplication on the t-variable. We also get a projection π of X_Q down to $\mathbb{P}^1$, by letting

$$\pi(x) := \frac{t^{-\eta+1} z^\alpha(x)}{t^{-\eta} z^\alpha(x)}$$

for any η, α such that this is well defined. Donaldson shows in [5] that if one excludes $\pi^{-1}(\infty)$, then the triple L_Q, ρ and π is in fact a test configuration, so π is flat and the fiber over 1 of (X_Q, L_Q) is isomorphic to (X_P, L_P).

It was shown by Lazarsfeld-Mustaţă in [11], Example 6.1, that if one choses the coordinates, or actually the flag of subvarieties, so that it is invariant under the torus action, the Okounkov body of a toric line bundle is equal to its defining polytope, up to translation. Thus we may assume that $P = \Delta(L_P)$ and

$$v(z^\alpha) = \alpha.$$

The invariant meromorphic extension of the section $z^\alpha \in H^0(kL_P)$ is $z^\alpha \in H^0(kL_Q)$, where we have identified X_P with the fiber over 1. By our calculations in Section 6, equation (6), the weight of $t^{-\eta} z^\alpha$ is η. Thus we see that

$$G_k(\alpha) = \sup\{\eta : t^{-\eta} z^{k\alpha} \in H^0(kL_Q)\} = kg(\alpha),$$

by the correspondence (13) and the fact that g is the defining equation for the roof of Q. We get that G_k/k is equal to the function g restricted to $\Delta_k(L)$, and thus by the convergence of G_k/k to $G[\mathcal{T}]$, that

$$G[\mathcal{T}] = g.$$

We see that our concave transform $G[\mathcal{T}]$ is a proper generalization of the well-known correspondence between test configurations and concave functions in toric geometry.

It is thus clear that, as was shown for product test configurations in Proposition 6.8, for toric test configurations it holds that the pushforward measure

$$G[\mathcal{T}]_* d\lambda_{|L_P} = g_* d\lambda_{|P}$$

is the sum of a piecewise polynomial measure and a multiple of a dirac measure, simply because P is a polytope and g is piecewise affine (the dirac measure part coming the top of the roof).

8 Deformation to the normal cone

One interesting class of test configurations is the ones which arise as a deformation to the normal cone with respect to some subscheme. This is described in detail by Ross-Thomas in [17] and [18], and we will only give a brief outline here.

Let Z be any proper subscheme of X. Consider the blow up of $X\times\mathbb{C}$ along $Z\times\{0\}$, and denote it by $\mathcal{X}$. Hence we get a projection π to $\mathbb{C}$ by composition $\mathcal{X}\to X\times\mathbb{C}\to\mathbb{C}$. We let P denote the exceptional divisor, and for any positive rational number c we get a line bundle

$$\mathcal{L}_c := \pi^* L - cP.$$

By Kleimans criteria (see e.g. [10]) it follows that $\mathcal{L}_c$ is relatively ample for small c. The action on $(X\times\mathbb{C}, L\times\mathbb{C})$ given by multiplication on the $\mathbb{C}$-coordinate lifts to an action ρ on $(\mathcal{X},\mathcal{L}_c)$, since both $Z\times\{0\}$ and $L\times\mathbb{C}$ are invariant under the action downstairs. Ross-Thomas in [17] show that this data defines a test configuration.

From the proof of Theorem 4.2 in [17] we get that

$$H^0(\mathcal{X}, k\mathcal{L}_c) = \bigoplus_{i=1}^{ck} t^{ck-i} H^0(X, kL\otimes\mathcal{J}_Z^i)\oplus t^{ck}\mathbb{C}[t]H^0(kL), \tag{14}$$

for k sufficiently large and $ck\in\mathbb{N}$. Here $\mathcal{J}_Z$ denotes the ideal sheaf of Z, and the sections of kL are being identified with their invariant extensions. From the expression (14) we can read off the associated filtration $\mathcal{F}$ of $H^0(kL)$. That

$$t^{ck}H^0(kL)\subseteq H^0(\mathcal{X}, k\mathcal{L}_c)$$

means that

$$\mathcal{F}_{-ck}H^0(kL) = H^0(kL).$$

Furthermore, for $0\leq i\leq ck$ and any $s\in H^0(kL)$ we get that $t^{ck-i}s\in H^0(\mathcal{X},k\mathcal{L}_c)$ iff $s\in H^0(kL\otimes\mathcal{J}_Z^i)$. This implies that for $-ck\leq\eta\leq 0$,

$$\mathcal{F}_\eta H^0(kL) = H^0(kL\otimes\mathcal{J}_Z^{ck+\eta}).$$

Also, when $\eta>0$ we get that $\mathcal{F}_\eta H^0(kL)=\{0\}$. In summary, if we let $g_{c,k}$ be defined by

$$g_{c,k}(\eta) := \lceil \max(\eta+ck, 0)\rceil$$

for $\eta \in (-\infty, 0]$ and let $g_{c,k} \equiv \infty$ on $(0, \infty)$, then by our calculations

$$\mathcal{F}_\eta H^0(kL) = H^0(kL \otimes \mathcal{J}_Z^{g_{c,k}(\eta)}). \tag{15}$$

Thus this natural class of filtrations can be seen as coming from test configurations.

Let us assume that Z is an ample divisor with a defining holomorphic section $s \in H^0(Z)$, i.e. $Z = \{s = 0\}$. Let a be a number between zero and c, then $L - aZ$ is still ample. Using multiplication with s^{ka} we can embed $H^0(k(L - aZ))$ into $H^0(kL)$. With respect to this identification of $H^0(k(L - aZ))$ as a subspace of $H^0(kL)$ for all k, we can identify the Okounkov body of $L - aZ$ with a subset of $\Delta(L)$. By vanishing theorems (see e.g. [11]), for large k

$$H^0(k(L - aZ)) = H^0(kL \otimes \mathcal{J}_Z^{ka}), \tag{16}$$

and therefore by (15)

$$H^0(k(L - aZ)) = \mathcal{F}_{k(a-c)} H^0(kL).$$

It follows that the part of $\Delta(L)$ where $G[\mathcal{T}]$ is greater or equal to $a - c$ coincides with $\Delta(L - aZ)$.[1]

Recall that by Theorem 2.8

$$\mathrm{vol}_{\mathbb{R}^n} \Delta(L - aZ) = \frac{\mathrm{vol}(L - aZ)}{n!}.$$

By this, a direct calculation yields that the pushforward measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ can be written as

$$\frac{\mathrm{vol}(L - cZ)}{n!}\delta_0 - \chi_{[-c,0]}\frac{d}{dx}\left(\frac{\mathrm{vol}(L - (x + c)Z)}{n!}\right)dx,$$

where δ_0 denotes the dirac measure at zero and $\chi_{[-c,0]}$ the indicator function of the interval $[-c, 0]$. Since for any ample (or even nef) ample line bundle the volume is given by integration of the top power of the first Chern class,

$$\mathrm{vol}(L) = \int_X c_1(L)^n,$$

it follows that the volume function is polynomial of degree n in the ample cone. Thus the measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ is a sum of a polynomial measure of degree less than n and a dirac measure.

Let again Z be an arbitrary subscheme of X. Consider the blow up of X along Z, and let E denote the exceptional divisor. If E is irreducible we may introduce local holomorphic coordinates (z_i) on the blow up, such that locally E is given by the equation $z_1 = 0$. Using these coordinates we get an associated Okounkov body $\Delta(L)$. For $s \in H^0(kL)$, the first coordinate of $v(s)$ is equal to the vanishing order of s along Z, i.e. the largest integer r such that $s \in H^0(kL \otimes \mathcal{J}_Z^r)$. Thus by (15) we get that

$$\Delta_{k,\eta}(L) = \{v(s)/k : s \in \mathcal{F}_\eta H^0(kL)\} = \Delta_k(L) \cap \{x_1 \geq g_{c,k}(\eta)/k\}.$$

[1] We thank Julius Ross for poining this out to us.

Furthermore

$$G_k(\alpha) = \sup\{\eta : \alpha \in \Delta_{k,\eta}(L)\} =$$
$$= \sup\{\eta : \alpha_1 \geq g_{c,k}(\eta)/k\} = k\min(\alpha_1 - c, 0),$$

and therefore

$$G[\mathcal{T}](x) = \min(x_1 - c, 0).$$

9 Product test configurations and geodesic rays

There is an interesting interplay between on the one hand test configurations and geodesic rays in the space of metrics on the other (see e.g. [14] and [16]). The model case is when we have a product test configuration.

Let $\mathcal{H}_L$ denote the space of positive hermitian metrics ψ of a positive line bundle L over X. The tangent space of $\mathcal{H}_L$ at any point ψ is naturally identified with the space of smooth real-valued functions on X. The works of Mabuchi, Semmes and Donaldson (see [12], [19] and [4]) have shown that there is a natural Riemannian metric on $\mathcal{H}_L$, by letting the norm of a tangent vector u at a point $\psi \in \mathcal{H}_L$ be defined by

$$||u||_\psi^2 := \int_X |u|^2 dV_\psi,$$

where $dV_\psi := (dd^c\psi)^n$. Let ψ_t be a ray of metrics, $t \in (0, \infty)$. We may extend it to complex valued t in $\mathbb{C}^\times$ if we let ψ_t be independent on the argument of t. We say that ψ_t is a geodesic ray if

$$(dd^c\psi_t)^{n+1} = 0 \tag{17}$$

on $X \times \mathbb{C}^\times$. The equation (17) is the geodesic equation with respect to the Riemannian metric on $\mathcal{H}_L$ (see e.g. [16]).

Let $\mathcal{T}$ be a product test configuration. That means that there is a $\mathbb{C}^\times$-action ρ on the original pair (X, L). Restriction of ρ to the unit circle gives a S^1-action. Let φ be an S^1-invariant positive metric on L. We get a $\mathbb{C}^\times$ ray $\tau \longmapsto \varphi_\tau \in \mathcal{H}_L$ of metrics by letting for any $\xi \in L$

$$|\xi|_{\varphi_\tau} := |\rho(\tau)^{-1}\xi|_\varphi. \tag{18}$$

Similarly we get corresponding rays $k\varphi_\tau$ in $\mathcal{H}_{kL}$. Since φ was assumed to be S^1-invariant, φ_τ only depends on the absolute value $|\tau|$. Also because the action ρ is holomorphic, it follows that

$$(dd^c\varphi_\tau)^{n+1} = 0,$$

therefore φ_τ is a geodesic ray.

In [1] Berndtsson introduces sequences of spectral measures on $\mathbb{R}$ arising naturally from a geodesic segment of metrics, and shows that they converge weakly to a certain pushforward of a volume form on X. Inspired by his result, we consider the analogue in our setting.

Let $\dot{\varphi}$ denote the derivative of φ_τ at 1, so $\dot{\varphi}$ is a smooth real-valued function on X. We consider the positive measure on $\mathbb{R}$ we get by pushing forward the volume form $dV_\varphi := (dd^c\varphi)^n$ on X with this function divided by two,

$$\mu_\varphi := (\dot{\varphi}/2)_* dV_\varphi.$$

The measure μ_φ does not does not depend on the choice of S^1-invariant metric φ. In fact, we have the following result.

Theorem 9.1. *Let $G[\mathcal{T}]$ denote the concave transform of the product test configuration. We have an equality of measures*

$$\mu_\varphi = G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

Proof. We will use one of the main ideas in the proof of the main result of Berndtsson in [1], Theorem 3.3. However, in our setting where the geodesic comes from a $\mathbb{C}^\times$-action things are much simpler since we do not need the powerful estimates used in [1].

Let dV be some fixed smooth volume form on X. We will introduce two families of scalar products on $H^0(kL)$, parametrized by τ, $||.||_{\tau,1}$ and $||.||_{\tau,2}$. First we let for any $s \in H^0(kL)$

$$||s||^2_{\tau,1} := \int_X |s|^2_{k\varphi_\tau} dV,$$

while we let

$$||s||^2_{\tau,2} := \int_X |\rho(\tau)^{-1}s|^2_{k\varphi} dV = ||\rho(\tau)^{-1}s||^2_{1,1}.$$

Direct calculations yield that

$$\frac{d}{d\tau}||s||^2_{\tau,1} = \frac{d}{d\tau}\int_X |s|^2_{k\varphi_\tau} dV = \int_X (-k\dot{\varphi}_\tau)|s|^2_{k\varphi_\tau} dV = (T_{-k\dot{\varphi}_\tau}s, s)_{\tau,1}, \tag{19}$$

where $T_{-k\dot{\varphi}_\tau}$ denotes the Toeplitz operator with symbol $-k\dot{\varphi}_\tau$.

Differentiating $||.||_{\tau,2}$ with respect to τ we get that

$$\frac{d}{d\tau}||s||^2_{\tau,2} = \frac{d}{d\tau}(\rho(\tau)^{-1}s, \rho(\tau)^{-1}s)_{1,1} = ((\frac{d}{d\tau}\rho(\tau)^{-2})s, s)_{1,1}. \tag{20}$$

On the other hand

$$\begin{aligned} ||s||^2_{\tau,1} = \int_X |s(x)|^2_{k\varphi_\tau} dV(x) = \int_X |\rho(\tau)^{-1}(s(x))|^2_{k\varphi} dV(x) = \\ = \int_X |(\rho(\tau)^{-1}s)(x)|^2_{k\varphi} dV(\rho(\tau)x) = \int_X |\rho(\tau)^{-1}s|^2_{k\varphi} dV_\tau, \end{aligned} \tag{21}$$

where $dV_\tau(x) := dV(\rho(\tau)x)$ thus denotes the resulting volume form after the τ-action. Since $dV_\tau(x)$ depends smoothly on τ, using (21) we get that

$$\begin{aligned} \left|\frac{d}{d\tau}_{|\tau=1}||s||^2_{\tau,1} - \frac{d}{d\tau}_{|\tau=1}||s||^2_{\tau,2}\right| = \left|\frac{d}{d\tau}_{|\tau=1}\int_X |\rho(\tau)^{-1}s|^2_{k\varphi}(dV_\tau - dV)\right| \leq \\ \leq \int_X |\frac{d}{d\tau}_{|\tau=1} dV_\tau| \int_X |s|^2_{k\varphi} dV = C||s||^2_{1,1}, \end{aligned} \tag{22}$$

where thus C is a uniform constant independent of s and k. Therefore letting $\tau = 1$ in equations (19) and (20), and using (22) we get that

$$\frac{d}{d\tau}\rho(\tau)_{|\tau=1} = T_{k\dot{\varphi}/2} + E_k, \tag{23}$$

where the error term E_k is uniformly bounded, $||E_k|| < C'$.

Let A be a self-adjoint operator on a N-dimensional Hilbert space, and let λ_i denote the eigenvalues of A, which therefore are real, counted with multiplicity. The spectral measure of A, denoted by $\nu(A)$, is defined as

$$\nu(A) := \sum_i \delta_{\lambda_i}.$$

We consider the normalized spectral measure of $T_{k\dot{\varphi}/2}$,

$$\nu_k := \frac{1}{k^n}\nu(T_{k\dot{\varphi}/2}/k).$$

By Theorem 3.2 in [1], which is a variant of a theorem of Boutet de Monvel-Guillemin (see [3]), we get that the measures ν_k converge weakly as k tends to infinity to the measure μ_φ.

Let $H^0(kL) = \sum_\eta V_\eta$ be the decomposition in weight spaces, and let P_η denote the projection to V_η. Then

$$\rho(\tau) = \sum_\eta \tau^\eta P_\eta,$$

and thus

$$\frac{d}{d\tau}\rho(\tau)_{|\tau=1} = \sum \eta P_\eta. \tag{24}$$

From (24) we see that the normalized spectral measures of $\frac{d}{d\tau}\rho(\tau)_{|\tau=1}$, which we denote by μ_k, coincides with the previously defined weight measure

$$\tilde{\mu}(\mathcal{T}, k) = \frac{1}{k^n}\sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_{k^{-1}\eta}.$$

According to Theorem 6.5 the sequence $\tilde{\mu}(\mathcal{T}, k)$, and therefore μ_k, converges weakly to the measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$.

Lastly, by the the min-max principle, when perturbing an operator A by an operator E with small norm $||E|| < \varepsilon$, then each eigenvalue is perturbed at most by ε. Thus from (23) it follows that $\nu_k - \mu_k$ converges weakly to zero, and the theorem follows. $\square$

We will relate this result to our previous discussion on Duistermaat-Heckman measures in Section 5 and 6, by showing that the map $\dot{\varphi}/2$ is a Hamiltonian for the S^1-action when the symplectic form is given by $dd^c\varphi$. This is of course well-known (see e.g. [4]), but we include it here for the benefit of the reader.

Let V be the holomorphic vector field on X generating the action ρ. Hence, the imaginary part ImV of V generates the S^1-action. By definition, $\dot{\varphi}/2$ is a Hamiltonian if it holds that

$$\mathrm{Im}V \rfloor dd^c\varphi = d\dot{\varphi}/2, \tag{25}$$

where $\rfloor$ denotes the contraction operator.

If we can show that

$$-iV \rfloor dd^c \varphi = \bar{\partial} \dot{\varphi}/2,$$

equation (25) will follow by taking the real part on both sides. We calculate locally with respect to some trivialization and without loss of generality we may assume that

$$V = \frac{\partial}{\partial z_1}.$$

Recall that by definition

$$dd^c \varphi = \frac{i}{2} \sum \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Hence we get that

$$-iV \rfloor dd^c \varphi = \frac{1}{2} \sum \frac{\partial^2 \varphi}{\partial z_1 \partial \bar{z}_j} d\bar{z}_j = \frac{1}{2} \bar{\partial} \frac{\partial \varphi}{\partial z_1}.$$

Since $V = \partial/\partial z_1$ generates the action, it follows that locally $\partial/\partial z_1 \varphi = \dot{\varphi}$, and we are done.

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